Master Integrals for Fermionic Contributions
to Massless Three-Loop Form Factors

G. Heinrich[a], T. Huber[b,c] and D. Maître[d]

[a] *School of Physics, The University of Edinburgh, Edinburgh EH9 3JZ, Scotland, UK*

[b] *Institut für Theoretische Physik, Universität Zürich, Winterthurerstrasse 190,
CH-8057 Zürich, Switzerland*

[c] *Institut für Theoretische Physik E, RWTH Aachen
D-52056 Aachen, Germany*

[d] *Stanford Linear Accelerator Center, Stanford University, Stanford, CA 94309, USA*

Abstract

In this letter we continue the calculation of master integrals for massless three-loop form factors by giving analytical results for those integrals which are relevant for the fermionic contributions proportional to N_F^2, $N_F \cdot N$, and N_F/N. Working in dimensional regularisation, we express one of the integrals in a closed form which is exact to all orders in ϵ, containing Γ-functions and hypergeometric functions of unit argument. In all other cases we derive multiple Mellin-Barnes representations from which the coefficients of the Laurent expansion in ϵ are extracted in an analytical form. To obtain the finite part of the three-loop quark and gluon form factors, all coefficients through transcendentality six in the Riemann ζ-function have to be included.

1   Introduction

The quark form factor $\gamma^* \to q\bar{q}$ and gluon form factor $H \to gg$ (effective coupling) are the simplest processes containing infrared divergences at higher orders in massless quantum field theory, and therefore appear in a large variety of physical applications. They can for instance be used to predict the infrared pole structure of multi-leg amplitudes at a given order [1–3] and to extract resummation coefficients [4,5], and they are needed for the purely virtual corrections to a number of collider reactions (Drell-Yan process, Higgs production and decay, DIS).

At the two-loop level, corrections to the massless quark [6] and gluon [7,8] form factors were computed in dimensional regularisation with $D = 4 - 2\epsilon$ to order ϵ^0 and subsequently extended to all orders in ϵ in Ref. [9]. Two-loop corrections to this order were also obtained for massive quarks [10]. The three-loop form factors to order ϵ^{-1} (and ϵ^0 for fermion loop contributions) were computed in [5,11]. One of the main motivations for obtaining analytical results for the form factors is the search for a deeper underlying structure of the coefficients, as proposed in Ref. [12] for planar box amplitudes.

In order to calculate the quark and gluon form factors at higher orders in perturbation theory, the amplitudes are reduced, by means of algebraic reduction procedures [13–16], to a small set of master integrals. At the three-loop level, the master integrals for massless form factors were identified in Ref. [17] and results for certain subsets are available in the literature [13, 17–19]. Among the three-loop master integrals, the genuine three-loop vertex functions are the most challenging ones from a computational point of view. They correspond to two-particle cuts of the master integrals for massless four-loop off-shell propagator integrals [20], which have been used in the calculation of the scalar R-ratio [21]. The derivation of the three-loop vertex integrals is of comparable complexity to the four-loop propagator integrals.

Working in dimensional regularisation and expanding the master integrals in a Laurent series in ϵ, the finite part of the three-loop form factors requires the extraction of all coefficients through transcendentality six, i.e. coefficients containing terms up to π^6 or ζ_3^2. Note that the power of ϵ coming with coefficients of transcendentality six in the Laurent expansion is not always the same in the different master integrals: Transcendentality six can appear in the coefficients of the ϵ^0-, ϵ- or ϵ^2-terms in the Laurent series. If it appears in the ϵ^k-term, this indicates that a prefactor $\sim 1/\epsilon^k$ will come from the reduction to master integrals, such that an expansion up to transcendentality six of the master integrals will always be required.

Those genuine three-loop vertex functions which contain one-loop or two-loop propagator insertions were already given in Ref. [17]. The purpose of the present letter is to extend this calculation to all three-loop master integrals which have less than nine propagators. Each topology contains only one master integral, which is chosen to be the scalar integral, with no loop momenta in the numerator and with all propagators raised to unit power. It turns out that this subset of three-loop master integrals is sufficient in order to obtain the aforementioned fermion loop contributions within a Feynman diagrammatic approach [22]. At this point we would like to point out an error in Ref. [17], namely the basis of three-loop master integrals given there is too large, since certain two-particle cuts of four-loop propagator topologies [20] yield topologically identical three-loop vertex topologies. Consequently $A_{8,1} = A_{8,2} \equiv A_8$ and $A_{9,2} = A_{9,3}$. The corrected set of three-loop vertex integrals is given in Fig. 1.

This letter is organised as follows. Computational methods to obtain analytical and nu-

merical results of the three-loop vertex integrals with up to eight propagators are described in Section 2, and the analytical results for them are listed in Section 3. Section 4 contains our conclusions and an outlook.

2   Master integrals: Classification and computational methods

Vertex integrals with one off-shell and two on-shell legs and massless propagators depend only on one kinematic scale: the mass q^2 of the off-shell leg. The dependence on this scale is given by the mass dimension of the integral, such that the coefficients of the Laurent expansion in the dimensional regularisation parameter ϵ are real constants which are in general of increasing transcendentality in the Riemann ζ-function, where the degree of transcendentality (DT) is defined by

$$
\begin{aligned}
DT(r) &= 0 \quad \text{for rational } r \\
DT(\pi^k) &= DT(\zeta(k)) = k \\
DT(x \cdot y) &= DT(x) + DT(y) \,.
\end{aligned}
\tag{1}
$$

At the three-loop level the quark form factor depends – like the process $e^+e^- \to 3$ jets at NNLO – on the following seven colour structures [22, 23]

$$
N^2 \,, \qquad N^0 \,, \qquad 1/N^2 \,, \qquad N_F \cdot N \,, \qquad N_F/N \,, \qquad N_F^2 \,, \qquad N_{F,\gamma} \,,
\tag{2}
$$

where the last colour factor stems from topologies in which the external gauge boson couples to a closed fermion loop. The three terms containing N_F are referred to as *fermionic corrections*. They have been derived in Refs. [5, 11] from the behaviour of the three-loop deep inelastic coefficient functions [24]. In the more conventional approach of computing multi-loop Feynman amplitudes the form factors are – after an algebraic reduction procedure [13–16] – expressed in terms of a small set of master integrals. It turns out [22] that the master integrals in Fig. 1 with at most eight propagators are sufficient in order to obtain the fermionic corrections to the form factor. The purpose of this letter is therefore to evaluate these master integrals. Those master integrals in Fig. 1 that contain single or multiple bubble insertions have already been computed in Ref. [17], the remaining ones with up to eight propagators – i.e. diagrams $A_{6,2}$, $A_{7,3}$, $A_{7,4}$, $A_{7,5}$, and A_8 – are subject of the present work. Working in dimensional regularisation with $D = 4 - 2\epsilon$, we give one of the diagrams ($A_{7,4}$) in a closed form which is exact to all orders in ϵ, containing Γ-functions and hypergeometric functions of unit argument. In all other cases we derive multiple – twofold to fourfold – Mellin-Barnes representations [25–28] from which the coefficients of the Laurent expansion in ϵ are obtained in an analytic form. As explained above, all coefficients through transcendentality six in the Riemann ζ-function have to be included to obtain the finite part of the three-loop form factor.

For many practical applications, and to verify the analytical results, it is sufficient to know the numerical values of the coefficients in the Laurent expansion of the master integrals to some finite order. There are several techniques to obtain numerical values for the coefficients, one of them being the sector decomposition method, which is described in detail in Refs. [29, 30]. Using this technique, the Laurent expansions of all master integrals relevant to the three-loop

form factors can be computed to, in principle, any desired order. In practice there are of course limitations, from CPU time for the numerical evaluation and from memory for the algebraic part of the sector decomposition procedure. The eight propagator graph A_8 is the most complex one from the sector decomposition point of view, not only due to the high number of propagators, but also because it exhibits spurious linear divergences at intermediate stages, which render the subtractions and thus the functions to be integrated more complicated. The computing time for A_8 up to order ϵ for a numerical precision of 0.1% is of the order of 4 hours on a 3.0 GHz PC. For a precision of 1% the evaluation is more than 10 times faster.

Another method of doing numerical cross checks proceeds along the lines of deriving Mellin-Barnes representations of loop integrals by means of the package `AMBRE` [31] and subsequently performing the analytical continuation and numerical evaluation of the obtained expressions with the package `MB` [32]. This procedure allowed us to check most of the coefficients at the sub-permille level.

3 Results

In this section we list the results we obtained for the three-loop master integrals necessary for the fermionic corrections to the three-loop quark form factor, which are the diagrams $A_{6,2}$, $A_{7,3}$, $A_{7,4}$, $A_{7,5}$ and A_8 in Figure 1. All other diagrams with up to eight propagators possess so-called bubble insertions and have already been given in Ref. [17].

Diagram $A_{6,2}$

The first diagram to be considered is $A_{6,2}$. In Ref. [33] a representation of this diagram in terms of a one-dimensional integral over hypergeometric functions was given. Here we pursue a different strategy and derive a twofold Mellin-Barnes representation [26–28] from which the coefficients of the Laurent series expansion about $\epsilon = 0$ can be computed. We start with

$$A_{6,2} = \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \int \frac{d^D r}{(2\pi)^D} \; \frac{1}{(k+p_1)^2 \, (k+l-p_2)^2 \, l^2 \, r^2 \, (r-k)^2 \, (r-k-l)^2} \; , \quad (3)$$

and assume here and in the following that all propagators contain an infinitesimal $+i\eta$. We then derive the following expression that contains a triple integral over a Meijer-G function [34,35],

$$A_{6,2} = -i \, S_\Gamma^3 \left[-q^2 - i\eta \right]^{-3\epsilon} \frac{\Gamma^3(1-\epsilon)\,\Gamma(3\,\epsilon)}{\Gamma(1-2\,\epsilon)\,\Gamma(2-4\,\epsilon)}$$

$$\times \int_0^1 dx \, dy \, dz \; x^{-\epsilon} \, (1-x)^{-3\,\epsilon} \, y^{-\epsilon} \, (1-y)^{-3\,\epsilon} \, z^{-2\,\epsilon} \, (1-z)^{-2\,\epsilon}$$

$$\times G_{33}^{32} \left(x\,z + y\,(1-z) \, \middle| \, \begin{matrix} \{-1+4\,\epsilon, -1+4\,\epsilon\}\,, \{3\,\epsilon\} \\ \{-1+3\,\epsilon, -1+2\,\epsilon, 0\}\,, \{\} \end{matrix} \right) \; , \quad (4)$$

$$\text{where} \quad q^2 = (p_1+p_2)^2 \quad \text{and} \quad S_\Gamma = \frac{1}{(4\pi)^{D/2}\,\Gamma(1-\epsilon)} \; . \quad (5)$$
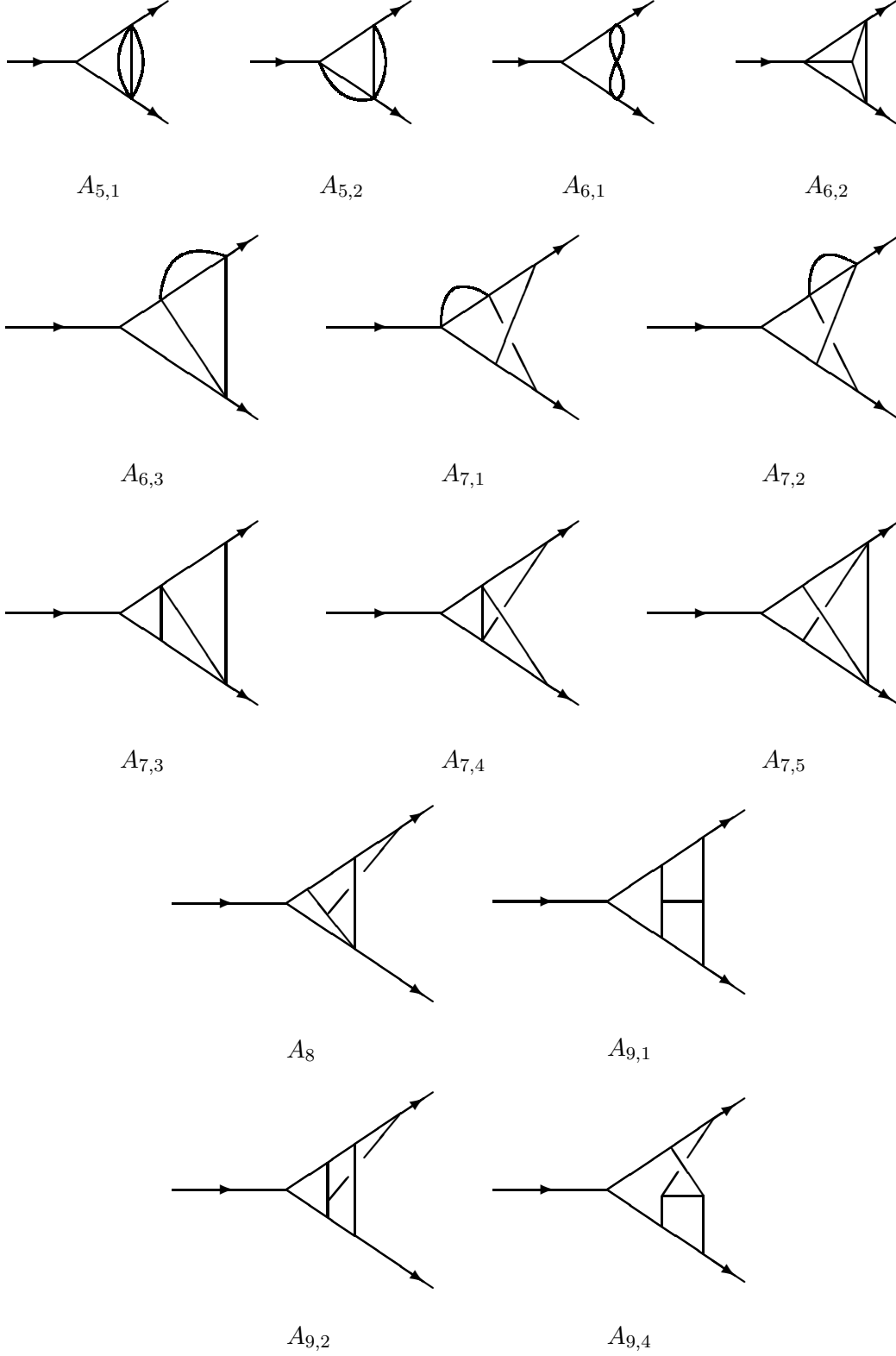
Figure 1: Three-loop master integrals with massless propagators. The incoming momentum is $q = p_1 + p_2$. Outgoing lines are considered on-shell and massless, i.e. $p_1^2 = p_2^2 = 0$.

4

We now make use of the contour integral representation of the Meijer-G function [34, 35], and subsequently decompose the argument by means of a second Mellin-Barnes representation. The integrals over x, y, and z can then be done in terms of Γ-functions. This leads to the following twofold Mellin-Barnes representation for $A_{6,2}$,

$$
A_{6,2} = -i\,S_\Gamma^3 \left[-q^2 - i\,\eta\right]^{-3\,\epsilon} \frac{\Gamma^3(1-\epsilon)\,\Gamma(3\,\epsilon)\,\Gamma^2(1-3\,\epsilon)}{\Gamma(1-2\,\epsilon)\,\Gamma(2-4\,\epsilon)} \int\limits_{c_1-i\infty}^{c_1+i\infty} \frac{dw_1}{2\pi i} \int\limits_{c_2-i\infty}^{c_2+i\infty} \frac{dw_2}{2\pi i}
$$

$$
\times\, \frac{\Gamma(-1+3\,\epsilon-w_1)\,\Gamma(-1+2\,\epsilon-w_1)\,\Gamma(2-4\,\epsilon+w_1)\,\Gamma(-w_2)\,\Gamma(w_2-w_1)}{\Gamma(3\,\epsilon-w_1)\,\Gamma(2-4\,\epsilon+w_2)\,\Gamma(2-4\,\epsilon+w_1-w_2)}
$$

$$
\times\, \Gamma(1-\epsilon+w_2)\,\Gamma(1-\epsilon+w_1-w_2)\,\Gamma(1-2\,\epsilon+w_2)\,\Gamma(1-2\,\epsilon+w_1-w_2)\ . \tag{6}
$$

In the above equation (6) the contour integrals in the complex plane are along curves which separate left poles of Γ-functions from right ones, where "left poles" are poles stemming from a $\Gamma(\ldots+w)$ dependence, while "right poles" stem from a $\Gamma(\ldots-w)$ dependence [28]. The most convenient choice for these contours are straight lines parallel to the imaginary axis, *i.e.* the real parts along the curves are constant. According to Refs. [26, 27], these real parts, together with the parameter ϵ, must be chosen in such a way as to have positive arguments in all occurring Γ-functions in order to separate left and right poles in the desired way. One verifies easily that

$$
c_1 = -\frac{6}{5}\,, \qquad c_2 = -\frac{1}{2}\,, \qquad -\frac{1}{15} < \epsilon < \frac{3}{20} \tag{7}
$$

is an appropriate choice in Eq. (6). From the fact that the origin lies within the allowed region for ϵ, we conclude that the Mellin-Barnes integration does not produce any poles in ϵ in addition to the UV pole that is already present in the prefactor. Therefore the expansion in ϵ commutes with the contour integrations. Proceeding in this way, the Mellin-Barnes integrations can be done order by order in ϵ. During this procedure, the contours can be closed at infinity to either side of the complex plane, and the corresponding residues are then summed with the appropriate global sign. Furthermore, the nested sums algorithm [36, 37] and the formulas in the Appendix of Ref. [28] – Barnes Lemmata and corollaries thereof – prove extremely useful. The final result for $A_{6,2}$ is

$$
A_{6,2} = i\,S_\Gamma^3 \left[-q^2 - i\,\eta\right]^{-3\,\epsilon}
$$

$$
\times \left[-\frac{2\,\zeta_3}{\epsilon} - 18\,\zeta_3 - \frac{7\pi^4}{180} + \left(-122\,\zeta_3 - \frac{7\pi^4}{20} + \frac{2\pi^2}{3}\,\zeta_3 - 10\,\zeta_5\right)\epsilon \right.
$$

$$
\left. + \left(-738\,\zeta_3 - \frac{427\pi^4}{180} + 6\pi^2\,\zeta_3 - 90\,\zeta_5 + \frac{163\pi^6}{7560} + 76\,\zeta_3^2\right)\epsilon^2 + \mathcal{O}(\epsilon^3) \right]\ . \tag{8}
$$

In Ref. [33], two more orders of the ϵ-expansion can be found.

Diagram $A_{7,3}$

We now turn our attention to the integral $A_{7,3}$. This integral will be represented, similarly

to the integral $A_{6,2}$, as a multiple Mellin-Barnes integral:

$$
\begin{aligned}
A_{7,3} &= \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \int \frac{d^D r}{(2\pi)^D} \; \frac{1}{k^2 \, (k+q)^2 \, (l-k-p_2)^2 \, (l-p_2)^2 \, (r+l)^2 \, r^2 \, (r-p_1)^2} \\
&= i\, S_\Gamma^3 \left[-q^2 - i\,\eta \right]^{-1-3\epsilon} \frac{\Gamma^4(1-\epsilon)\,\Gamma(-\epsilon)}{\Gamma(1-2\,\epsilon)\,\Gamma(1-3\,\epsilon)} \int\limits_{c_1-i\infty}^{c_1+i\infty} \frac{dw_1}{2\pi i} \int\limits_{c_2-i\infty}^{c_2+i\infty} \frac{dw_2}{2\pi i} \int\limits_{c_3-i\infty}^{c_3+i\infty} \frac{dw_3}{2\pi i} \; \frac{\Gamma(-w_1)}{\Gamma(1-w_1)} \\
&\quad \times \frac{\Gamma(-3\,\epsilon - w_3)\,\Gamma(1+2\,\epsilon+w_1+w_2)\,\Gamma(1+w_1+w_2)\,\Gamma(-2\,\epsilon-w_2)\,\Gamma(-\epsilon-w_1)}{\Gamma(1-3\,\epsilon-w_3)\,\Gamma(2-2\,\epsilon+w_1+w_2)} \\
&\quad \times \Gamma(-w_3)\,\Gamma(\epsilon-w_1-w_2+w_3)\,\Gamma(1-\epsilon+w_2)\,\Gamma(1+w_3)\,\Gamma(-\epsilon+w_1-w_3) \; . \qquad (9)
\end{aligned}
$$

The contour integrals are again along straight lines in the complex plane parallel to the imaginary axis, and as before we must choose the real parts of the integration variables such as to have positive arguments in all occurring Γ-functions. This is achieved by choosing

$$
c_1 = -\frac{3}{20}\,, \qquad c_2 = -\frac{3}{5}\,, \qquad c_3 = -\frac{1}{2}\,, \qquad -\frac{1}{8} < \epsilon < \frac{3}{20}. \qquad (10)
$$

As it was the case for $A_{6,2}$, we have the origin within the allowed region for ϵ and therefore the Mellin-Barnes integration does not give rise to any additional poles in ϵ, the only pole of the integral being the infrared pole that is already present in the prefactor in Eq. (9). We can thus again perform the contour integrations order by order in ϵ. Since the leading coefficient turns out to have already transcendentality five, we only need to compute the first two terms in the expansion. They are given by

$$
A_{7,3} = i\, S_\Gamma^3 \left[-q^2 - i\,\eta \right]^{-1-3\epsilon} \left[\left(-\frac{\pi^2\,\zeta_3}{6} - 10\,\zeta_5 \right) \frac{1}{\epsilon} - \frac{119\,\pi^6}{2160} - \frac{31}{2}\,\zeta_3^2 + \mathcal{O}(\epsilon) \right]. \qquad (11)
$$

During the evaluation of this integral we could not proceed until the end by merely applying Barnes Lemmata and corollaries thereof, but had to apply auxiliary integral representations of hypergeometric functions at intermediate steps. The benefit of this procedure is that it enables us to perform all Mellin-Barnes integrations, at the cost of introducing additional parameters over which we subsequently have to integrate. However, the integrations over these auxiliary parameters can be done in terms of logarithms and (harmonic) polylogarithms. Furthermore, we made extensive use of the package HPL [38,39] and of an algorithm based on the nested sums approach [36,37]. See Appendix A for more details on this point.

Diagram $A_{7,4}$

The next diagram we consider is $A_{7,4}$. At first glance it looks quite difficult since it lacks both a bubble insertion and a planar topology. However, it turns out to be simpler than the planar diagram $A_{7,3}$, and it can even be displayed in a closed form. The main reason for this is the fact that at the outer vertices of both outgoing lines only three lines meet, and hence the introduction of Feynman parameters allows for the combination of propagators that differ only

by a light-like momentum. This property is absent in both $A_{6,2}$ and $A_{7,3}$, both of which did not reveal a closed form but only a multiple Mellin-Barnes representation. For $A_{7,4}$, we find

$$
\begin{aligned}
A_{7,4} &= \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \int \frac{d^D r}{(2\pi)^D} \frac{1}{k^2 (k-q)^2 (r+l-k)^2 l^2 (l-p_1)^2 r^2 (r-p_2)^2} \\
&= i\, S_\Gamma^3 \left[-q^2 - i\eta \right]^{-1-3\epsilon} \cdot 2 \cdot \Gamma^4(1-\epsilon)\, \Gamma^2(-\epsilon) \\
&\quad \times \Bigg[\frac{\Gamma(1-\epsilon)\,\Gamma(3\epsilon)}{(1-3\epsilon)^2\, \Gamma(2-4\epsilon)}\; _4F_3(1, 1-\epsilon, 1-3\epsilon, 2-6\epsilon; 2-3\epsilon, 2-3\epsilon, 2-4\epsilon; 1) \\
&\qquad - \frac{\Gamma(1-3\epsilon)\,\Gamma(2-3\epsilon)\,\Gamma(3\epsilon)\,\Gamma(1+2\epsilon)}{\Gamma(2-\epsilon)\,\Gamma(2-6\epsilon)} \\
&\qquad \times\, _4F_3(1, 1, 1+2\epsilon, 2-3\epsilon; 2, 2, 2-\epsilon; 1) \\
&\qquad + \frac{\Gamma^2(1-3\epsilon)\,\Gamma(1+2\epsilon)\,\Gamma(1+3\epsilon)}{\Gamma(2-\epsilon)\,\Gamma(2-6\epsilon)} \\
&\qquad \times\, _4F_3(1, 1, 1+2\epsilon, 1+3\epsilon; 2, 2, 2-\epsilon; 1) \Bigg] \\
&= i\, S_\Gamma^3 \left[-q^2 - i\eta \right]^{-1-3\epsilon} \\
&\quad \times \Bigg[\frac{6\zeta_3}{\epsilon^2} + \left(\frac{11\pi^4}{90} + 36\zeta_3 \right) \frac{1}{\epsilon} + \left(\frac{11\pi^4}{15} + 216\zeta_3 - 2\pi^2 \zeta_3 + 46\zeta_5 \right) \\
&\qquad + \left(\frac{22\pi^4}{5} - \frac{19\pi^6}{270} + 1296\zeta_3 - 12\pi^2\zeta_3 - 282\zeta_3^2 + 276\zeta_5 \right) \epsilon + \mathcal{O}(\epsilon^2) \Bigg], \qquad (12)
\end{aligned}
$$

where the expansion in ϵ was done by means of the `Mathematica` [40] package `HypExp` [33, 41].

Diagram $A_{7,5}$

We now consider diagram $A_{7,5}$ for which we derive the following fourfold Mellin-Barnes representation.

$$
\begin{aligned}
A_{7,5} &= \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \int \frac{d^D r}{(2\pi)^D} \frac{1}{k^2 (k+q)^2 (k+r)^2 (l-p_2)^2 (r-l)^2 r^2 (k+l+p_1)^2} \\
&= i\, S_\Gamma^3 \left[-q^2 - i\eta \right]^{-1-3\epsilon} \frac{\Gamma^3(1-\epsilon)}{\Gamma(1-2\epsilon)\,\Gamma(1-4\epsilon)} \int_{c_1-i\infty}^{c_1+i\infty} \frac{dw_1}{2\pi i} \int_{c_2-i\infty}^{c_2+i\infty} \frac{dw_2}{2\pi i} \int_{c_3-i\infty}^{c_3+i\infty} \frac{dw_3}{2\pi i} \int_{c_4-i\infty}^{c_4+i\infty} \frac{dw_4}{2\pi i} \\
&\qquad \frac{\Gamma(w_4-w_1)\,\Gamma(1+w_3)\,\Gamma(-3\epsilon-w_3)\,\Gamma(1-2\epsilon+w_1+w_2-w_4)\,\Gamma(-w_3)\,\Gamma(-w_4)}{\Gamma(1-w_1+w_3+w_4)}
\end{aligned}
$$

7

$$\times \frac{\Gamma(1+3\epsilon+w_3+w_4)\,\Gamma(1+\epsilon+w_1+w_2)\,\Gamma(1+w_1+w_2)\,\Gamma(-\epsilon-w_1)\,\Gamma(-\epsilon-w_2)}{\Gamma(2+\epsilon+w_1+w_2)\,\Gamma(2-2\epsilon+w_1+w_2)}$$

$$\times \frac{\Gamma(1-\epsilon+w_2)\,\Gamma(1+w_3)\,\Gamma(1-\epsilon+w_1)\,\Gamma(\epsilon-w_1-w_2+w_3+w_4)\,\Gamma(w_4-w_2)}{\Gamma(1-w_2+w_3+w_4)} \quad .$$

$$(13)$$

Like in the previous cases, the Mellin-Barnes integral does not generate poles in ϵ, so we can therefore interchange the expansion in ϵ with the contour integrations. We choose

$$c_1 = -\frac{1}{5}\,, \qquad c_2 = -\frac{1}{4}\,, \qquad c_3 = -\frac{1}{7}\,, \qquad c_4 = -\frac{1}{11}. \qquad (14)$$

As before in the case of $A_{7,3}$ we can not proceed until the end by merely applying Barnes Lemmata and corollaries thereof, but again have to apply auxiliary integral and series representations at intermediate steps, this time even for a larger class of functions than before. Besides hypergeometric functions, these are mainly logarithms and (harmonic) polylogarithms as well as ψ-functions with

$$\psi^{(0)}(z) \;=\; \frac{d}{dz}\ln\left[\Gamma(z)\right]\,,$$

$$\psi^{(k)}(z) \;=\; \frac{d}{dz}\psi^{(k-1)}(z) \quad \text{for} \quad k=1,2,\dots \qquad . \qquad (15)$$

The sums and integrals over the auxiliary parameters are then performed by means of the same techniques as before. Some details can again be found in Appendix A. The final result for $A_{7,5}$ reads

$$A_{7,5} = i\,S_\Gamma^3\left[-q^2-i\,\eta\right]^{-1-3\,\epsilon}\left[2\pi^2\,\zeta_3 + 10\,\zeta_5 + \left(12\pi^2\,\zeta_3 + 60\,\zeta_5 + \frac{11\pi^6}{162} + 18\,\zeta_3^2\right)\epsilon + \mathcal{O}(\epsilon^2)\right].$$

$$(16)$$

Diagram A_8

The last diagram we consider is A_8 which can also be displayed as a fourfold Mellin-Barnes integral.

$$A_8 \;=\; \int\frac{d^Dk}{(2\pi)^D}\int\frac{d^Dl}{(2\pi)^D}\int\frac{d^Dr}{(2\pi)^D}\,\frac{1}{(k+p_1)^2\,(k+r)^2\,(k+r+q)^2\,(l-k)^2\,(l+r)^2\,l^2\,r^2\,(l+p_1)^2}$$

$$=\; -i\,S_\Gamma^3\left[-q^2-i\,\eta\right]^{-2-3\,\epsilon}\frac{\Gamma^3(1-\epsilon)\,\Gamma(-1-3\epsilon)}{\Gamma(-2\,\epsilon)\,\Gamma(-4\,\epsilon)}\int\limits_{c_1-i\infty}^{c_1+i\infty}\frac{dw_1}{2\pi i}\int\limits_{c_2-i\infty}^{c_2+i\infty}\frac{dw_2}{2\pi i}\int\limits_{c_3-i\infty}^{c_3+i\infty}\frac{dw_3}{2\pi i}\int\limits_{c_4-i\infty}^{c_4+i\infty}\frac{dw_4}{2\pi i}$$

$$\frac{\Gamma(1+w_3)\,\Gamma(1+w_4)\,\Gamma(w_4-w_2)\,\Gamma(w_3-w_1)\,\Gamma(-w_4)\,\Gamma(-w_3)\,\Gamma(2+w_1+w_2)}{\Gamma(2+w_3+w_4)\,\Gamma(1+w_4-w_2)\,\Gamma(1+w_3-w_1)}$$

$$\times \frac{\Gamma(2+\epsilon+w_1+w_2)\,\Gamma(1-\epsilon+w_1)\,\Gamma(1-\epsilon+w_2)\,\Gamma(-1-\epsilon-w_1)\,\Gamma(-1-\epsilon-w_2)}{\Gamma(2-2\epsilon+w_1+w_2)\,\Gamma(3+\epsilon+w_1+w_2)}$$

$$\times \Gamma(2+3\epsilon+w_3+w_4)\,\Gamma(1-2\epsilon+w_1+w_2-w_3-w_4)\,\Gamma(\epsilon-w_1-w_2+w_3+w_4) \ . \tag{17}$$

This time the Mellin-Barnes integral does indeed generate poles in ϵ. We choose [32]

$$c_1 = -\frac{7}{8}\,, \qquad c_2 = -\frac{19}{24}\,, \qquad c_3 = -\frac{13}{24}\,, \qquad c_4 = -\frac{25}{48}\,, \qquad -\frac{5}{16} < \epsilon < -\frac{5}{24} \tag{18}$$

in order to separate left poles of Γ-functions from right ones, and subsequently perform the analytic continuation to $\epsilon = 0$ [32]. This generates four kernels, one four-dimensional one, two three-dimensional ones, and one two-dimensional one. We arrive at the final result

$$A_8 \;=\; i\,S_\Gamma^3 \left[-\,q^2 - i\,\eta\right]^{-2-3\epsilon} \left[\frac{8\zeta_3}{3\epsilon^2} + \left(\frac{5\pi^4}{27} - 8\zeta_3\right)\frac{1}{\epsilon} + 24\zeta_3 - \frac{5\pi^4}{9} - \frac{52}{9}\,\pi^2\zeta_3 + \frac{352}{3}\,\zeta_5\right.$$

$$\left. + \left(-72\zeta_3 + \frac{5\pi^4}{3} + \frac{52}{3}\,\pi^2\zeta_3 - 352\,\zeta_5 + \frac{1709\pi^6}{8505} - \frac{332}{3}\zeta_3^2\right)\epsilon + \mathcal{O}(\epsilon^2)\right]. \tag{19}$$

Despite the fact that this integral has one more propagator compared to $A_{7,5}$ it was much simpler to evaluate than the former one, and we did not have to introduce any auxiliary parameters in integral or series representations, but could proceed until the end by the same techniques as described in $A_{6,2}$. The reason for the simplicity of A_8 is again that it possesses – contrary to $A_{7,5}$ – an outgoing line with an outer vertex where only three lines meet.

4 Conclusions and Outlook

In this letter we have evaluated those master integrals for massless three-loop form factors which are necessary for the calculation of the fermionic corrections to the quark form factor. We obtained analytical results for all coefficients through transcendentality six in the Riemann ζ-function, as required to obtain the finite part of the form factor at the three-loop level. For the integral $A_{7,4}$ we could obtain a representation which is valid to all orders in ϵ, in terms of hypergeometric functions of unit argument. For the other integrals, we derived multiple Mellin-Barnes representations from which we extracted all necessary coefficients order by order in ϵ.

The only missing pieces to complete the set of master integrals for massless three-loop form factors are therefore the three diagrams in Figure 1 which have nine propagators. It turns out that each of them can be expressed in terms of a sixfold Mellin-Barnes representation [42] which gives rise to $\mathcal{O}(100)$ single terms upon performing the analytical continuation to $\epsilon = 0$ by means of the package MB [32]. Although the number of single integrals is quite large due to the extraction of high poles in ϵ, and the evaluation is not completely automated at certain stages, the analytical results for these integrals are within reach [42].

9

Acknowledgements

A   Some technical details on performing MB integrals

This appendix is devoted to some details about those terms in diagrams $A_{7,3}$ and $A_{7,5}$ which require the introduction of auxiliary parameters in addition to the MB variables. Consider the term

$$\int\limits_{c_2-i\infty}^{c_2+i\infty} \frac{dw_2}{2\pi i} \int\limits_{c_3-i\infty}^{c_3+i\infty} \frac{dw_3}{2\pi i} \left(-\frac{3\pi^2 \csc^2(\pi w_2)\Gamma(w_2+1)\Gamma(1-w_3)\Gamma(-w_3)\Gamma^3(w_3)}{(w_2+1)\Gamma(w_2+w_3+2)} \right) \tag{20}$$

which appears in the computation of $A_{7,3}$ at the stage where two MB integrations are left (the first MB integration of this integral can be done by Barnes Lemma and corollaries thereof, see [28]). The c_i are as in Eq. (10), and we have converted certain combinations of Γ-functions to Csc.

We perform the integration over w_3 by closing the contour to the right. After summing all residues, one obtains

$$\int\limits_{c_2-i\infty}^{c_2+i\infty} \frac{dw_2}{2\pi i} \quad \Big[\frac{\pi^2 \csc^2(\pi w_2)[\psi^{(0)}(w_2+2)]^3}{2\,(w_2+1)^2} + \frac{5\pi^4 \csc^2(\pi w_2)\psi^{(0)}(w_2+2)}{4\,(w_2+1)^2}$$

$$+\frac{3\pi^2 \csc^2(\pi w_2)[\psi^{(0)}(w_2+2)+\gamma_E][\gamma_E\psi^{(0)}(w_2+2)+\psi^{(1)}(w_2+2)]}{2\,(w_2+1)^2}$$

$$-\frac{\pi^2 \csc^2(\pi w_2)\psi^{(2)}(w_2+2)}{(w_2+1)^2} + \frac{\pi^2 \csc^2(\pi w_2)[2\gamma_E^3+5\gamma_E\pi^2+4\zeta_3]}{4\,(w_2+1)^2}$$

$$+\frac{3\pi^2 \csc^2(\pi w_2)\,{}_4F_3(1,1,1,1\,;\,2,2,w_2+3\,;\,1)}{(w_2+1)^2(w_2+2)} \Big], \tag{21}$$

where γ_E is Euler's constant. The integration of all but the last term in the above equation can be carried out with the standard technique of the nested sums algorithm [36, 37]. In the last term we write the hypergeometric function as

$$\int\limits_0^1 dt\, \frac{(w_2+2)(1-t)^{w_2+1} Li_2(t)}{t} \,. \tag{22}$$

The integration over w_2 can now be done in this term as well. In the end, the integration over the auxiliary parameter t can be carried out by means of the package HPL [38, 39]. This yields $9\zeta_3^2 + 349\pi^6/15120$ for the expression in Eq. (20). An alternative approach would be to replace in Eq. (20)

$$\frac{\Gamma(w_2 + 1)}{\Gamma(w_2 + w_3 + 2)} = \frac{1}{\Gamma(w_3 + 1)} \int\limits_0^1 dt \, t^{w_2} \, (1 - t)^{w_3} \tag{23}$$

with the benefit of having factorized the integrand in w_2 and w_3. The integrations over w_2 and w_3 can now be carried out by summation of appropriate residues, followed by integration over t.

As far as $A_{7,5}$ is concerned, we consider the finite piece of Eq. (13) where ϵ is set to zero. We perform the integration over w_4 and close the contour to the right. The result contains two hypergeometric functions which we decompose as follows

$$_4F_3(-w_1, -w_2, w_3 + 1, -w_1 - w_2 + w_3 \, ; \, -w_1 - w_2, -w_1 + w_3 + 1, -w_2 + w_3 + 1 \, ; \, 1) =$$

$$\int_0^1 dt_1 \int_0^1 dt_2 \int_0^1 dt_3 \, \frac{\Gamma(-w_2 + w_3 + 1)\Gamma(-w_1 + w_3 + 1)\Gamma(-w_1 - w_2)}{\Gamma(-w_1)\Gamma(-w_2)\Gamma(-w_1 - w_2 + w_3)\Gamma^2(1 + w_3)\Gamma(-w_3)}$$

$$\times \frac{t_1^{-w_1-1}(1 - t_1)^{w_3} \, t_2^{-w_2-1}(1 - t_2)^{w_3} \, t_3^{-w_1-w_2+w_3-1}(1 - t_3)^{-w_3-1}}{(1 - t_1 t_2 t_3)^{w_3+1}} \, , \tag{24}$$

$$_4F_3(w_1 + 1, w_2 + 1, w_3 + 1, w_1 + w_2 + w_3 + 2 \, ; \, w_1 + w_2 + 2, w_1 + w_3 + 2, w_2 + w_3 + 2 \, ; \, 1) =$$

$$\int_0^1 dt_1 \int_0^1 dt_2 \int_0^1 dt_3 \, \frac{\Gamma(w_1 + w_2 + 2)\Gamma(w_1 + w_3 + 2)\Gamma(w_2 + w_3 + 2)}{\Gamma(w_1 + 1)\Gamma(w_2 + 1)\Gamma(w_1 + w_2 + w_3 + 2)\Gamma^2(1 + w_3)\Gamma(-w_3)}$$

$$\times \frac{t_1^{w_1}(1 - t_1)^{w_3} \, t_2^{w_2}(1 - t_2)^{w_3} \, t_3^{w_1+w_2+w_3+1}(1 - t_3)^{-w_3-1}}{(1 - t_1 t_2 t_3)^{w_3+1}} \, . \tag{25}$$

We then perform the integration over w_3, followed by the other two MB integrations. In the end, we carry out the integrations over t_1, t_2 and t_3. We make simple variable changes in the t_1-t_2-t_3 cube where appropriate.

References

[1] L. Magnea and G. Sterman, Phys. Rev. D **42** (1990) 4222.

[2] S. Catani, Phys. Lett. B **427** (1998) 161 [hep-ph/9802439].

[3] G. Sterman and M.E. Tejeda-Yeomans, Phys. Lett. B **552** (2003) 48 [hep-ph/0210130].

[4] L. Magnea, Nucl. Phys. B **593** (2001) 269 [arXiv:hep-ph/0006255].

[5] S. Moch, J.A.M. Vermaseren and A. Vogt, JHEP **0508** (2005) 049 [hep-ph/0507039].

[6] G. Kramer and B. Lampe, Z. Phys. C **34** (1987) 497; **42** (1989) 504(E);
T. Matsuura and W.L. van Neerven, Z. Phys. C **38** (1988) 623;
T. Matsuura, S.C. van der Maarck and W.L. van Neerven, Nucl. Phys. B **319** (1989) 570.

[7] R.V. Harlander, Phys. Lett. B **492** (2000) 74 [hep-ph/0007289].

[8] V. Ravindran, J. Smith and W.L. van Neerven, Nucl. Phys. B **704** (2005) 332 [hep-ph/0408315].

[9] T. Gehrmann, T. Huber and D. Maître, Phys. Lett. B **622** (2005) 295 [hep-ph/0507061].

[10] W. Bernreuther *et al.*, Nucl. Phys. B **706** (2005) 245 [hep-ph/0406046]; Nucl. Phys. B **712** (2005) 229 [hep-ph/0412259]; Nucl. Phys. B **723** (2005) 91 [hep-ph/0504190].

[11] S. Moch, J.A.M. Vermaseren and A. Vogt, Phys. Lett. B **625** (2005) 245 [hep-ph/0508055].

[12] Z. Bern, L. J. Dixon and V. A. Smirnov, Phys. Rev. D **72** (2005) 085001 [arXiv:hep-th/0505205].

[13] F.V. Tkachov, Phys. Lett. **100B** (1981) 65;
K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. **B192** (1981) 159.

[14] S. Laporta, Int. J. Mod. Phys. A **15** (2000) 5087 [hep-ph/0102033].

[15] T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [hep-ph/9912329].

[16] C. Anastasiou and A. Lazopoulos, JHEP **0407** (2004) 046 [hep-ph/0404258].

[17] T. Gehrmann, G. Heinrich, T. Huber and C. Studerus, Phys. Lett. B **640**, 252 (2006) [arXiv:hep-ph/0607185].

[18] S. Bekavac, Comput. Phys. Commun. **175** (2006) 180 [arXiv:hep-ph/0505174].

[19] S.G. Gorishnii, S.A. Larin, L.R. Surguladze and F.V. Tkachov, Comput. Phys. Comm. **55** (1989) 381;
S.A. Larin, F.V. Tkachov and J.A.M. Vermaseren, NIKHEF-H-91-18.

[20] P.A. Baikov, Phys. Lett. B **634** (2006) 325 [hep-ph/0507053].

[21] P.A. Baikov, K.G. Chetyrkin and J.H. Kühn, Phys. Rev. Lett. **96** (2006) 012003 [hep-ph/0511063].

[22] T. Gehrmann and B. Tödtli, private communication

[23] A. Gehrmann-De Ridder, T. Gehrmann, E. W. N. Glover and G. Heinrich, JHEP **0711** (2007) 058. [arXiv:0710.0346 [hep-ph]].

[24] J.A.M. Vermaseren, A. Vogt and S. Moch, Nucl. Phys. B **724** (2005) 3 [hep-ph/0504242].

[25] V.A. Smirnov, Phys. Lett. B **460** (1999) 397 [hep-ph/9905323].

[26] J.B. Tausk, Phys. Lett. B **469** (1999) 225 [hep-ph/9909506].

[27] C. Anastasiou and A. Daleo, JHEP **0610** (2006) 031 [arXiv:hep-ph/0511176].

[28] V. A. Smirnov, *"Evaluating Feynman integrals"*, Springer Tracts Mod. Phys. **211** (2004) 1.

[29] T. Binoth and G. Heinrich, Nucl. Phys. B **585** (2000) 741 [hep-ph/0004013].

[30] T. Binoth and G. Heinrich, Nucl. Phys. B **680** (2004) 375 [hep-ph/0305234].

[31] J. Gluza, K. Kajda and T. Riemann, Comput. Phys. Commun. **177** (2007) 879 [arXiv:0704.2423 [hep-ph]].

[32] M. Czakon, Comput. Phys. Commun. **175** (2006) 559 [arXiv:hep-ph/0511200].

[33] T. Huber and D. Maître, arXiv:0708.2443 [hep-ph].

[34] A. Erdélyi (ed.), Higher transcendental functions, Vol. 1, (McGraw-Hill, New York, 1953) .

[35] A. Erdélyi (ed.), Tables of Integral Transforms, Vol. 2, (McGraw-Hill, New York, 1954) .

[36] J. A. M. Vermaseren, Int. J. Mod. Phys. A **14** (1999) 2037 [arXiv:hep-ph/9806280].

[37] S. Moch, P. Uwer and S. Weinzierl, J. Math. Phys. **43** (2002) 3363 [arXiv:hep-ph/0110083].

[38] D. Maître, Comput. Phys. Commun. **174** (2006) 222 [arXiv:hep-ph/0507152].

[39] D. Maître, arXiv:hep-ph/0703052.

[40] *MATHEMATICA* 5.2, Copyright 2005 by Wolfram Research.

[41] T. Huber and D. Maître, Comput. Phys. Commun. **175** (2006) 122 [hep-ph/0507094].

[42] T. Huber et. al., work in progress